

THE L^2 WEAK SEQUENTIAL CONVERGENCE OF RADIAL MASS CRITICAL NLS SOLUTIONS WITH MASS ABOVE THE GROUND STATE.

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ABSTRACT. We study the non-scattering L^2 solution u to the radial mass critical nonlinear Schrödinger equation with mass just above the ground state, and show that there exists a time sequence $\{t_n\}_n$, such that $u(t_n)$ weakly converges to the ground state Q up to scaling and phase transformation. We also give some partial results on the mass concentration of the minimal mass blow up solution.

1. INTRODUCTION

In this work, we study the Cauchy Problem to the focusing mass critical NLS at regularity L^2

$$\begin{cases} iu_t + \Delta u = -|u|^{4/d}u, \\ u(0, x) = u_0 \in L^2(\mathbb{R}^d). \end{cases} \quad (1.1)$$

Here d denotes the dimension. We assume $d = 1, 2, 3$ to reduce technicalities, but the results should hold for general dimension as well.

Equation (1.1) has three conservation laws:

- Mass:

$$M(u(t, x)) := \int |u(t, x)|^2 dx = M(u_0), \quad (1.2)$$

- Energy:

$$E(u(t, x)) := \frac{1}{2} \int |\nabla u(t, x)|^2 dx - \frac{1}{2 + \frac{4}{d}} \int |u(t, x)|^{2 + \frac{4}{d}} dx = E(u_0), \quad (1.3)$$

- Momentum:

$$P(u(t, x)) := \Im \left(\int \nabla u(t, x) \overline{u(t, x)} dx \right) = P(u_0), \quad (1.4)$$

and the following symmetries:

- (1) Space-time translation: If $u(t, x)$ solves (1.1), then $\forall t_0 \in \mathbb{R}, x_0 \in \mathbb{R}^d$, we have $u(t - t_0, x - x_0)$ solves (1.1).
- (2) Phase transformation: If u solves (1.1), then $\forall \theta_0 \in \mathbb{R}$, we have $e^{i\theta_0} u$ solves (1.1).
- (3) Galilean transformation: If $u(t, x)$ solves (1.1), then $\forall \xi \in \mathbb{R}^d$, we have $u(t, x - \xi t) e^{i\frac{\xi}{2}(x - \frac{\xi}{2}t)}$ solves (1.1).

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- (4) Scaling: If $u(t, x)$ solves (1.1), then $\forall \lambda \in \mathbb{R}_+$, we have $u_\lambda(t, x) := \frac{1}{\lambda^{\frac{d}{2}}} u(\frac{t}{\lambda}, \frac{x}{\lambda})$ solves (1.1).
- (5) Pseudo-conformal transformation: If $u(t, x)$ solves (1.1), then $\frac{1}{t^{\frac{d}{2}}} \bar{u}(\frac{1}{t}, \frac{x}{t}) e^{i \frac{|x|^2}{4t}}$ solves (1.1).

Throughout the whole article, we assume

$$\|Q\|_2 \leq \|u_0\|_2 \leq \|Q\|_2 + \alpha. \quad (1.5)$$

Here α is some universal small constant, which will be determined later and Q is the unique L^2 , positive solution to the elliptic PDE

$$-\Delta Q + Q = |Q|^{4/d} Q. \quad (1.6)$$

We remark Qe^{it} solves (1.1) if and only if Q solves (1.6). Q is smooth and decays exponentially.

1.1. Main Results. In this work we show the following:

Theorem 1.1. *Assume u is a radial solution to (1.1), with u_0 satisfying (1.5), which does not scatter forward. Let $(T^-(u), T^+(u))$ be its lifespan, then there exist a time sequence $t_n \rightarrow T^+(u)$, and a family of parameters $\lambda_{*,n}, \gamma_{*,n}$ such that*

$$\lambda_{*,n}^{d/2} u(t_n, \lambda_{*,n} x) e^{-i\gamma_{*,n}} \rightharpoonup Q \text{ in } L^2. \quad (1.7)$$

See Definition 2.4 for the precise notion of scattering forward.

If one further assume $\|u\|_2 = \|Q\|_2$, then we can upgrade the above to

Theorem 1.2. *Assume u is a radial solution to (1.1), with $\|u\|_2 = \|Q\|_2$, which does not scatter forward. Let $(T^-(u), T^+(u))$ be its lifespan, then there exists a sequence $t_n \rightarrow T^+$, and a family of parameters $\lambda_{*,n}, \gamma_{*,n}$ such that*

$$\lambda_{*,n}^{d/2} u(t_n, \lambda_{*,n} x) e^{-i\gamma_{*,n}} \rightarrow Q \text{ in } L^2. \quad (1.8)$$

Remark 1.3. *Most of the proof for Theorem 1.1 and Theorem 1.2, written in this work can be obtained for the nonradial case as well. Indeed, only one step (Lemma 4.12) cannot be generalized to the nonradial case. In particular, we do not use Sobolev embedding or weighted Strichartz estimate for radial solutions. Moreover, our results hold in fact for solutions which are symmetric across any d linearly independent hyperplanes. Nevertheless, the idea in this work is not enough to cover the nonradial case. We will investigate this case in a future work.*

We also obtain some partial results for the minimal mass blow up solution to (1.1) at regularity L^2 , not necessarily radial.

Theorem 1.4. *Let u be a general L^2 solution to (1.1) that blows up at finite time T and such that $\|u(t)\|_2 = \|Q\|_2$. Then there exist sequences x_n, t_n such that*

$$\int_{|x-x_n| \leq (T-t_n)^{2/3-}} |u|^2 \geq \|Q\|_2. \quad (1.9)$$

We will give the proof of Theorem 1.4 in Appendix A.

Remark 1.5. *If one further assumes that the initial data is in H^1 and with same mass as Q , then (1.9) holds even if one changes the power $2/3$ to 1 . Indeed, the H^1 minimal mass blow up solution is determined by [M⁺93] and can be written down explicitly.*

Remark 1.6. *Though not explicitly stated in the literature, it is not hard to combine concentration compactness and Dodson's scattering result in [Dod15], (see also Theorem 2.8), to show that if u is a general solution to the (1.1), which blows up in finite time T , then there exist $t_n \rightarrow T$, and $x_n \in \mathbb{R}^d$, such that*

$$\int_{|x-x_n| \leq (T-t_n)^{1/2-}} |u|^2 \geq \|Q\|_2^2. \quad (1.10)$$

However, as far as we are concerned, it is always of interest to improve the above estimate to at least

$$\int_{|x-x_n| \leq (T-t_n)^{1/2}} |u|^2 \geq \|Q\|_2^2. \quad (1.11)$$

Estimate (1.11), for $u_0 \in H^1$ and satisfying (1.5), should follow using a rigidity result due to Raphaël [Rap05], which is highly nontrivial.

1.2. Background. Equation (1.1) is called focusing since its associated energy is not coercive, and it is called mass critical since its mass conservation law is invariant under scaling symmetry. Both focusing and defocusing NLS are locally well posed in L^2 , [CW89], see also [Caz03], [Tao06]. And it is an active research topic to understand the long time dynamic.

It is known that the solution to the defocusing NLS with arbitrary L^2 data does not break down and scatters to some linear solution, [Dod12].

On the other hand, the focusing problem, (1.1), is known to have more complicated dynamics, and the solution may break down in finite time, [Gla77]. It is of great interest to understand the blow up phenomena, rather than just showing the existence of blow up. We remark that since we are working on the critical space L^2 and pseudo conformal symmetry¹ is a symmetry in L^2 , finite time blow up solutions and non-scattering solutions are essentially the same.

We recall that the ground state Q gives a threshold of scattering dynamic. Indeed, it is known that any solution to (1.1) scatters to a linear solution if the initial data has mass strictly below the ground state, [Dod15]. See also previous work [Wei83].

The main purpose of this article is to understand the possible long time dynamic of (1.1) for solution with mass at or just above the threshold $\|Q\|_2^2$, hence assumption (1.5).

Under assumption (1.5), the finite time blow up² solution to (1.1) at regularity H^1 has been extensively studied in recent years. We recall the work of [LPSS88][Per01], [MR⁺05b], [MR03], [MR06], [MR04], [Rap05], [MR05a] regarding the so-called log-log blow up dynamic. If one assumes the initial data is in H^1 , with negative energy and satisfies assumption (1.5), then one can upgrade the sequential convergence in Theorem 1.1 to convergence as $t \rightarrow T^+$, [MR04]. It is also shown in [MR04] that for general H^1 solution to (1.1) satisfying (1.5), without the sign condition in the energy, Theorem 1.1 also holds. Regarding Theorem 1.2, if one

¹This symmetry, however, is not a symmetry in H^1 .

²Note that finite time blow up should be understood as one of the long time dynamics rather than a short time dynamic.

assumes the initial data is in H^1 and blows up in finite time, then the convergence holds as $t \rightarrow T^+$ thanks to Merle's complete classification of minimal mass blow up solutions. Moreover, in Theorem 1.2, if one assumes u is radial and in H^1 , $d = 2, 3$, and u is global, $\|u\|_2 = \|Q\|_2$, then one can still obtain convergence as $t \rightarrow T^+$, due to [LZ12]. [LZ12] indeed shows such solution must be a solitary wave. When assumes $d \geq 4$, (our work mainly deals with $d = 1, 2, 3$), if one assumes u is radial and in L^2 , $\|u\|_2 = \|Q\|_2$, global in both sides, then [LZ10] shows such solution must be a solitary wave.

The main purpose of this work is to extend these results, (except the 4 dimensional result [LZ10]), to the lower L^2 regularity.

We point out if one only wants to show some sequential weak convergence but does not want to characterize the limit profile, then assumption (1.5) or radial assumption may not be necessary, and the method in [DKM15] should be applicable to prove these kinds of results. Indeed, it is pointed out in [DKM15], their methods and strategy should be able to handle general dispersive equations once a suitable profile decomposition is available. However, assumption (1.5) and the radial assumption are very important for us to determine that the limit profile is indeed Q .

Results with type similar to Theorem 1.1, Theorem 1.2 will also appear naturally when one consider the mass concentration phenomena of finite time blow up solutions to (1.1), we refer to [MT90], [Caz03], [Naw90], [HK05], [CRSW04], [Tzi06], [VZ07], [Bou98], [Ker06], [BV07] and reference in their works.

Finally, we point out that our work is also motivated by the recent progress in the soliton resolution conjecture for the energy critical wave, (where more complete and general results are available), see [DKM11], [DKM12b], [DKM13], [DKM16c], [DKM12a], [DKM15], [CKLS14], [DKM16a], [DJKM16], [DKM16b] and the references therein.

1.3. Notation. Throughout this work, α is used to denote a universal small number, $\delta(\alpha)$ is a small number depending on α such that $\lim_{\alpha \rightarrow 0} \delta(\alpha) = 0$. We use C to denote a large constant, it usually changes line by line.

We write $A \lesssim B$ when $A \leq CB$, for some universal constant C , we write $A \gtrsim B$ if $B \lesssim A$. We write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. As usual, $A \lesssim_\sigma B$ means that $A \leq C_\sigma B$, where C_σ is a constant depending on σ .

We use the usual functional spaces L^p , we will also use the Sobolev space H^1 . We sometimes write L^p for $L^p(\mathbb{R}^d)$, and similarly for the other spaces. We also use $L_t^q L_x^p$ to denote $L^q(\mathbb{R}; L^p(\mathbb{R}^d))$. When a certain function is only defined on $I \times \mathbb{R}^d$, we also use the notation $L^q(I; L^p(\mathbb{R}^d))$. Sometimes we use $\|f\|_p$ to denote $\|f\|_{L^p}$. We also use $\|u\|_2$ to denotes the $\|u(0)\|_2$, when u is a solution to the Schrödinger equation with initial data u_0 , since L^2 is preserved under Schrödinger flow.

We use the standard Littlewood-Paley projection operator $P_{<N}, P_{>N}$, we quickly recall the definition here. Let ψ be a bump function which equals to 1 when $|x| \leq 1$, and vanishes for $|x| \geq 2$, one define the multiplier $P_{<N}$ as

$$\widehat{P_N f}(\xi) := \psi\left(\frac{\xi}{N}\right) \hat{f}(\xi). \quad (1.12)$$

And $P_{>N} = 1 - P_{<N}$.

We use \langle, \rangle to denote the usual L^2 (complex) inner product.

Finally, for a solution $u(t, x)$, we use $(T^-(u), T^+(u))$ to denote its lifespan.

2. PRELIMINARY

We present the preliminary for this work, experts may skip this section in the reading.

2.1. Local well posedness (LWP) and stability.

2.1.1. *Classical Strichartz estimates.* The local well posedness of (1.1) is established using the classical Strichartz Estimates. We recall them below. Consider the linear Schrödinger equation:

$$\begin{cases} iu_t + \Delta u = 0, \\ u(0, x) = u_0 \in L^2(\mathbb{R}^d). \end{cases} \quad (2.1)$$

We use $e^{it\Delta}$ to denote the linear propagator. One has estimates

$$\|e^{it\Delta}u_0\|_{L_{t,x}^{2(d+2)/d}} \lesssim \|u_0\|_{L_x^2}, \quad (2.2)$$

$$\left\| \int_0^t e^{i(t-s)\Delta} f(s, x) ds \right\|_{L_{t,x}^{2(d+2)/d} \cap L^\infty L^2} \lesssim \|f\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}}. \quad (2.3)$$

We refer to [Caz03] [KT98], [Tao06] and reference therein for a proof.

Remark 2.1. *Strichartz estimate holds in more general case, indeed one has*

$$\left\| \int_0^t e^{i(t-s)\Delta} f(s, x) ds \right\|_{L_t^q L_x^r} \lesssim \|f\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} \quad (2.4)$$

where (q, r) , (\tilde{q}, \tilde{r}) is admissible in the sense $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$ and $(q, r, d), (\tilde{q}, \tilde{r}, d) \neq (2, \infty, 2)$. We fix $(q, r) = (\tilde{q}, \tilde{r}) = (\frac{2(d+2)}{d}, \frac{2(d+2)}{d})$ for simplicity. Similarly, the local well posedness and stability holds in more general sense.

In the rest of this section, we quickly recall the classical results in the local well posedness theory without proof. We refer to [CW88], [CW90], [Tao06], [Caz03] and the reference there in for a proof.

2.1.2. Local existence.

Theorem 2.2. *Given u_0 in L^2 , there exists $T = T(u_0)$ such that there is a unique solution $u(t, x)$ to (1.1) with initial data u_0 in the following sense:*

$$u(t, x) = e^{it\Delta}u_0 + i \int_0^t e^{i(t-\tau)\Delta}(|u|^{4/d}u(\tau))d\tau, \quad t \in [0, T] \quad (2.5)$$

Formula (2.5) holds in space $C([0, T]; L^2) \cap L_{t,x}^{2(d+2)/d}([0, T] \times \mathbb{R}^d)$.

2.1.3. Blow up criteria. We have the following blow up criteria.

Proposition 2.3. *Given a solution u with initial data u_0 , assume (T_-, T_+) is the maximal time interval u can be defined on, if T_+ is finite, then*

$$\|u\|_{L^{2(d+2)/d}[0, T_+] \times \mathbb{R}^d} = \infty \quad (2.6)$$

Similarly results holds for T_- .

Now, we can define the notion of scattering.

Definition 2.4. *We say a solution u to (1.1) scatters forward if $T^+ = \infty$ and $\|u\|_{L^{2(d+2)/d}[0, T_+] \times \mathbb{R}^d} < \infty$. Similarly, we define the notion of scattering backward. If u scatters both backward and forward, then we say u scatters.*

2.1.4. *Small data theory.*

Proposition 2.5. *There exists $\epsilon_0 > 0$ such that if the initial data $u_0 \in L^2$ and satisfy*

$$\|e^{it\Delta}u_0\|_{L_{t,x}^{2(d+2)/d}} \leq \epsilon_0 \quad (2.7)$$

then the solution to (1.1) with initial data u_0 is global and one has estimate

$$\|u(t, x)\|_{L_{t,x}^{2(d+2)/d}} \lesssim \|u_0\|_2. \quad (2.8)$$

Remark 2.6. *By Strichartz estimate (2.2), it is clear that (2.7) holds when $\|u_0\|_2$ is small enough.*

2.1.5. *Stability.* We state a stability result about (1.1). This kind of argument is standard nowadays. One may refer to Lemma 3.9, lemma 3.10 in [CKS⁺08].

Proposition 2.7. *Let I be a compact interval, and $0 \in I$. Let \tilde{u} is a near-solution to (1.1) in the sense*

$$i\tilde{u}_t + \Delta\tilde{u} + |\tilde{u}|^{4/d}\tilde{u} = e, \quad (2.9)$$

and the following estimate holds

$$\|\tilde{u}\|_{L^{2(d+2)/d}(I \times \mathbb{R})} \leq M, \quad (2.10)$$

$$\|\tilde{u}\|_{L_T^\infty L^2(\mathbb{R})} \leq E, \quad (2.11)$$

$$\|e\|_{L^{\frac{2(d+2)}{d+4}}(I \times \mathbb{R})} \leq \epsilon, \quad (2.12)$$

where $\epsilon < \epsilon_1 := \epsilon_1(M_1, M_2)$. Assume further there exists $t_0 \in I$ and $u_0 \in L^2$ such that

$$\|\tilde{u}(t_0) - u_0\|_2 < \epsilon, \quad (2.13)$$

Then, there exists a unique solution $u(t, x)$ to the Cauchy Problem

$$\begin{cases} iu_t + \Delta u + |u|^{4/d}u = 0, \\ u(t_0) = u_0. \end{cases} \quad (2.14)$$

such that

$$\|u - \tilde{u}\|_{L^{\frac{2(d+2)}{d}}(I \times \mathbb{R}) \cap L_T^\infty L^2(\mathbb{R})} \lesssim_{M_1, M_2} \epsilon. \quad (2.15)$$

In particular

$$\|u\|_{L^{2(d+2)/d}(I \times \mathbb{R})} \lesssim_{M_1, M_2} 1. \quad (2.16)$$

2.2. Scattering below the mass of the ground state. The dynamic of (1.1) for initial data with mass blow up the ground state is known, due to the following Theorem by Dodson, [Dod15].

Theorem 2.8 (Dodson). *Consider the Cauchy Problem (1.1), with the initial data u_0 such that*

$$\|u_0\|_{L_x^2} < \|Q\|_{L_x^2}. \quad (2.17)$$

The solution u to (1.1) is global, further more it scatters

$$\|u(t, x)\|_{L_{t,x}^{2(d+2)/d}} < \infty. \quad (2.18)$$

2.3. Concentration compactness. Strichartz estimates (2.2) lacks compactness due to the symmetry of equation (2.1). Note the aforementioned symmetries for the nonlinear equation (1.1) also hold for the linear equation. Profile decomposition is the tool to remedy this.

Let us start with the following definition,

Definition 2.9. Let $G := \{g = g_{x_0, t_0, \xi_0, \lambda_0} | x_0, t_0, \xi_0 \in \mathbb{R}, \lambda_0 \in \mathbb{R}_+\}$, where $g_{x_0, \xi_0, \lambda_0, \xi_0}$ is a map $L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$:

$$g_{x_0, \xi_0, \lambda_0, t_0} f(x) := \frac{1}{\lambda_0^{\frac{d}{2}}} e^{ix\xi_0} (e^{i(-\frac{t_0}{\lambda_0^2})\Delta} f) \left(\frac{x - x_0}{\lambda_0} \right). \quad (2.19)$$

Remark 2.10. G is a group acting on L^2 and for any $g \in G, f \in L^2$, $\|g \cdot f\|_2 = \|f\|_2$.

2.3.1. Profile decomposition. One has

Proposition 2.11 (Theorem 5.4 [BV07], Theorem 4.2 [TVZ08], Theorem 2 [MV98]). Let $\{u_n\}_{n=1}^\infty$ be bounded in $L^2(\mathbb{R}^d)$, then up to extracting subsequence, there exist a family of L^2 functions ϕ_j , $j = 1, 2, \dots$ and group elements $g_{j,n} \in G$, where $g_{j,n} = (g_{j,n})_{x_{j,n}, \xi_{j,n}, \lambda_{j,n}, t_{j,n}}$, such that for all $l = 1, 2, \dots$ we have the decomposition

$$u_n = \sum_{j=1}^l g_{j,n} \phi_j + \omega_n^l, \quad (2.20)$$

(here (2.20) defines ω_n^l .) And the following properties hold

- *Asymptotically orthogonality of the group elements:* For any $j \neq j'$, one has

$$\frac{\lambda_{j,n}}{\lambda_{j',n}} + \frac{\lambda_{j',n}}{\lambda_{j,n}} + \left| \frac{x_{j,n} - x_{j',n}}{\lambda_{j,n}} \right| + |\lambda_{j,n}(\xi_{j,n} - \xi_{j',n})| + \left| \frac{t_{j,n} - t_{j',n}}{\lambda_{j,n}^2} \right| \xrightarrow{n \rightarrow \infty} \infty \quad (2.21)$$

- *Asymptotically orthogonality of mass:*

$$\forall l \geq 1, \lim_{n \rightarrow \infty} |M(u_n) - \sum_{j \leq l} M(\phi_j) - M(\omega_n^l)| = 0. \quad (2.22)$$

- *Asymptotically orthogonality of Strichartz norm:*

$$\forall j \neq j', \lim_{n \rightarrow \infty} \|e^{it\Delta}(g_{j,n}\phi_j)e^{it\Delta}(g_{j',n}\phi_{j'})\|_{L_{t,x}^{(d+2)/d}} = 0. \quad (2.23)$$

- *Smallness of remainder term:*

$$\lim_{l \rightarrow \infty} \limsup_{n \rightarrow \infty} \|e^{it\Delta}\omega_n^l\|_{L_{t,x}^{2(d+2)/d}} = 0 \quad (2.24)$$

- *Weak limit condition of the remainder term:*

$$\forall j \leq l, g_{j,n}^{-1}\omega_n^l \rightharpoonup 0 \text{ in } L_x^2 \quad (2.25)$$

According to Proposition 2.11, we define

Definition 2.12. A linear profile is a function $f \in L^2$ and a sequence $\{g_n\}_n \subset G$, or equivalently a function $f \in L^2$ with parameters $\{x_n, \xi_n, \lambda_n, t_n\}_n \subset \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}$.

Up to extracting subsequence and adjusting the profile, for every profile decomposition as in Proposition 2.11, we always assume without loss of generality that $\lim_{n \rightarrow \infty} -\frac{t_{j,n}}{\lambda_{j,n}^2} = 0$ or equal to $\pm\infty$. This leads to the following standard definition:

Definition 2.13. We call a profile f with parameter $\{x_n, \xi_n, \lambda_n, t_n\}_n$

- Compact profile if $t_n \equiv 0$,
- Forward scattering profile if $-\frac{t_n}{\lambda_n^2} = \infty$,
- Backward scattering profile if $-\frac{t_n}{\lambda_n^2} = -\infty$.

2.3.2. *Nonlinear approximation.* To deal with the nonlinear equation (1.1), one needs the notion of nonlinear profile.

Definition 2.14. Given a linear profile, i.e. a function $f \in L^2$ and a sequence $\{g_n\}_n \subset G$, or equivalently a function $f \in L^2$ with parameters $\{x_n, \xi_n, \lambda_n, t_n\}_n \subset \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}$. We say U is the nonlinear profile associated with this linear profile if U is a solution to nonlinear Schrödinger equation

$$iU_t + \Delta U + |U|^{4/d}U = 0, \quad (2.26)$$

and satisfy the estimate

$$\lim_{n \rightarrow \infty} \left\| U\left(-\frac{t_n}{\lambda_{j,n}^2}\right) - e^{i(-\frac{t_n}{\lambda_n^2})\Delta} f \right\|_2 = 0. \quad (2.27)$$

(2.27) is only required for n large enough. For (2.27) to make sense, we require U is defined in a neighborhood³ of $\lim_{n \rightarrow \infty} -\frac{t_n}{\lambda_n^2}$.

Remark 2.15. Given a linear profile, the associated nonlinear profile exists and is unique. The existence and uniqueness of the nonlinear profile basically relies on the local well posedness theory of (1.1). This is quite standard, see Notation 2.6 in [DKM11].

Now, we state a nonlinear approximation for NLS. Though we do not give the detailed proof here, we point out it is essentially the consequence of asymptotically orthogonality (2.21) and the classical stability theory Proposition 2.7, see [BV07], [MV98], [TVZ08] for more details.

Proposition 2.16. Assume that $\{u_n\}_n$ admits a profile decomposition with profiles $(\phi_j; \{x_{j,n}, \xi_{j,n}, \lambda_{j,n}, t_{j,n}\}_n)_j$ as in Proposition 2.11. Let us consider a sequence of Cauchy problems for a sequence of initial time $\{t_n\}$,

$$\begin{cases} i\partial_t v_n(t, x) + \Delta v_n(t, x) + |v_n|^4 v_n(t, x) = 0, \\ v_n(t_n, x) = u_n. \end{cases} \quad (2.28)$$

For each fixed j , let $\Phi_j(t, x)$ be the associated nonlinear profile to ϕ_j . Let

$$\Phi_{j,n} := \frac{1}{\lambda_{j,n}^{\frac{d}{2}}} e^{ix\xi_{j,n}} e^{-it|\xi_{j,n}|^2} \Phi_j \left(\frac{t - t_{j,n}}{\lambda_{j,n}^2}, \frac{x - x_{j,n} - 2\xi_{j,n}t}{\lambda_{j,n}} \right).$$

If for any $\{\tau_n\}_n$ such that:

$$\forall j > 0, \quad \Phi_j \text{ scatters forward or } \lim_{n \rightarrow \infty} \frac{\tau_n - t_{j,n}}{\lambda_{j,n}^2} < T_+(\Phi_j), \quad (2.29)$$

(One may extract a subsequence again so that the above limit exists),
then for n large enough, $v_n, \Phi_{j,n}$ are defined in $[t_n, t_n + \tau_n]$.

³In profile decomposition, one usually needs to extract subsequence many times, we always assume that $\lim_{n \rightarrow \infty} -\frac{t_n}{\lambda_n^2}$ exists or equal to $\pm\infty$. We also define the neighborhood of ∞ as (M, ∞) for any M , similarly we define the neighborhood of $-\infty$.

Moreover, let

$$r_n^l := v_n - \sum_{j=1}^l \Phi_{j,n} - e^{it_n \Delta} \omega_n^l, \quad (2.30)$$

then one has

$$\limsup_{l \rightarrow \infty} \limsup_{n \rightarrow \infty} \|r_n^l\|_{L^\infty([t_n, t_n + \tau_n]; L^2) \cap L^{2(d+2)/d}([t_n, t_n + \tau_n] \times \mathbb{R}^d)} = 0. \quad (2.31)$$

Furthermore, (2.21) implies that $\forall l > 0$

$$\begin{aligned} & \left\| \sum_{j=1}^l \Phi_{j,n} \right\|_{L^{2(d+2)/d}([t_n, t_n + \tau_n] \times \mathbb{R}^d)}^{\frac{2(d+2)}{d}} \\ & \lesssim \sum_{j=1}^l \|\Phi_j\|_{L^{2(d+2)/d}([\lim_{n \rightarrow \infty} \frac{-t_{j,n}}{\lambda_{j,n}^2}, \lim_{n \rightarrow \infty} \frac{\tau_n - t_{j,n}}{\lambda_{j,n}}] \times \mathbb{R}^d)}^{2(d+2)/d} + o_n(1) \end{aligned} \quad (2.32)$$

In particular, if for any j , Φ_j scatters forward, then for n large enough, the associated solution to (1.1) with initial data $u(t_n)$ also scatters forward.

We remark here similar results holds for energy critical wave and energy critical NLS, see [BG99], [Ker01].

2.4. Variational characterization of ground state. Let us first recall the classical Gagliardo-Nirenberg inequality.

Lemma 2.17. *Let $v \in H^1$, then*

$$E(v) \geq \frac{1}{2} \int |\nabla v|^2 \left[1 - \left(1 - \frac{\|v\|_2^2}{\|Q\|_2^2} \right)^{4/d} \right] \quad (2.33)$$

It gives the variational characterization of the ground state Q .

Lemma 2.18. *Let $v \in H^1$, and*

$$\int |v|^2 = \int Q^2, \quad E(v) = 0 \quad (2.34)$$

then

$$v(x) = \lambda_0^{N/2} Q(\lambda_0 x + x_0) e^{i\gamma_0}. \quad (2.35)$$

In [MR⁺05b], Merle and Raphaël apply concentration compactness type techniques to generalize the above into the following:

Lemma 2.19. *Let v in H^1 , there exists $\alpha_0 > 0$, such that for all $\alpha < \alpha_0$, there exists $\delta(\alpha) > 0$, such that if*

$$\|v\|_2 \leq \|Q\|_2 + \alpha, \quad E(v) \leq \alpha \|\nabla v\|_2^2, \quad (2.36)$$

then there exists $\lambda_0 = \frac{\|\nabla Q\|_2}{\|\nabla u\|_2}, x_0 \in \mathbb{R}^d, \gamma_0 \in \mathbb{R}$ such that

$$\left\| \frac{1}{\lambda_0^{d/2}} u\left(\frac{x - x_0}{\lambda_0}\right) e^{i\gamma_0} - Q \right\|_{H^1} \leq \delta(\alpha). \quad (2.37)$$

and, $\lim_{\alpha \rightarrow 0} \delta(\alpha) = 0$.

3. THE DYNAMIC OF NON-POSITIVE ENERGY SOLUTION

Throughout this section, we assume the solution u to (1.1) has H^1 initial data and satisfies (1.5).

The dynamics of non-positive energy solutions are extensively studied in the series work of Merle and Raphaël, [MR⁺05b], [MR03], [MR06], [MR04], [Rap05], [MR05a]. We will apply their results in this work. However, we will work from an L^2 based viewpoint rather than H^1 viewpoint.

Merle and Raphaël show all strictly negative energy solutions blows up according to the so-called log-log dynamic, we restate their results as the following,

Theorem 3.1. *Assume u is a solution to (1.1) with H^1 initial data, non positive energy, and satisfying assumption (1.5), assume further $\|u\|_2 \neq \|Q\|_2$ if u is of zero energy, then u blows up in finite time according to the so-called log-log law*

$$u(t, x) = \frac{1}{\lambda^{d/2}(t)} (Q + \epsilon) \left(\frac{x - x(t)}{\lambda} \right) e^{i\gamma(t)}, x(t) \in \mathbb{R}^d, \gamma(t) \in \mathbb{R}, \lambda(t) \in \mathbb{R}^+, \|\epsilon\|_{H^1} \leq \delta(\alpha) \quad (3.1)$$

with estimate

$$\lambda(t) \sim \sqrt{\frac{T-t}{\ln |\ln T-t|}}. \quad (3.2)$$

$$\lim_{t \rightarrow T} \int (|\nabla \epsilon(t, x)|^2 + \epsilon(t, x)^2 e^{-|x|}) = 0. \quad (3.3)$$

Remark 3.2. *We do not directly use (3.2). Our results rely on the fact such solution will blow up in finite time and the mechanism of blow up is ejecting mass out of the singular point, i.e. the control (3.3).*

One may refer to [MR⁺05b], [MR03], [MR06] for a full proof of Theorem 3.1 when the solution is of strictly negative energy. When the solution is of zero energy, one may refer to Theorem 3 in [MR06], see also Theorem 4 and Proposition 5 in [MR04]. For estimate (3.3), which is most relevant to our work, one may refer to the formula above (3.7) in page 52 of [MR06]. Strictly speaking, the term appears in [MR06] is \tilde{Q}_b rather than the ground state Q , but \tilde{Q}_b is just small modification of Q , and converges to Q in a strong way as $b \rightarrow 0$, see Proposition 1 in [MR06]. And $b \rightarrow 0$ as $t \rightarrow T$, see, again, the formula above (3.7) in page 52 of [MR06].

Now, for the purpose of our work, we write a corollary of Theorem 3.1.

Corollary 3.3. *Assume u is a solution to (1.1) with H^1 initial data, nonpositive energy, and satisfying assumption (1.5), assume further $\|u\|_2 \neq \|Q\|_2$ if u is of zero energy, then there exists $\delta = \delta(u) > 0$, such that $\forall A > 1$, there exists $T_1 < T^+(u)$, $x_1 \in \mathbb{R}^d, l_1 > 0$, such that*

$$\int_{|x-x_1| \leq l_1} |u(T_1, x)|^2 \geq \delta, \quad \int_{|x-x_1| \geq Al_1} |u(T_1, x)|^2 \geq \delta \quad (3.4)$$

See proof in Section 5.

4. AN OVERVIEW FOR THE PROOF FOR THEOREM 1.1, THEOREM 1.2

We give an overview for the proof of Theorem 1.1, Theorem 1.2. We mainly focus on the proof the Theorem 1.1. Indeed, Theorem 1.2 follows from Theorem 1.1 due to the following classical fact:

Lemma 4.1. *Let H be an Hilbert space, if $v_n \rightharpoonup v_0$ and $\|v_n\|_H = \|v\|_H$, then*

$$v_n \rightarrow v_0. \quad (4.1)$$

4.1. Introduction. Ever since the work of Kenig and Merle, [KM06], [KM08], there is a road map to approach results of type Theorem 1.1, which includes three ingredients:

- (1) concentration compactness theorems
- (2) variational characterization of ground state
- (3) rigidity theorems

Concentration compactness relies on the study of the **linear** operator $e^{it\Delta}$ and remedies the lack of compactness of classical Strichartz estimate caused by the symmetry of the system.

The concentration compactness will help us reduce the study of the original problem to the study of so-called almost periodic solution, i.e. solution of the form

$$u(t, x) = \frac{1}{\lambda^{d/2}(t)} P_t \left(\frac{x - x(t)}{\lambda(t)} \right) e^{ix\xi(t)}, \lambda(t) > 0, x(t), \xi(t) \in \mathbb{R}^d, \quad (4.2)$$

$$\{P_t\}_t \text{ precompact family in } L^2(\mathbb{R}^d).$$

Such strategy is also called Liouville Theorem in the literature, see for example, [MR04].

Variational characterization will help us understand why ground state Q is special. Thus, help us see the profile Q in the study of (4.2).

Rigidity theorem will tell us the so called almost periodic solution is special, and one may expect a powerful enough rigidity theorem should fully characterize solutions to (1.1) of type (4.2), though we cannot achieve this in this work.

4.2. Step 1: First extraction of profile. First, we will use the profile decomposition and a minimization argument in [DKM15] to show the following:

Lemma 4.2. *Let u be a solution, not necessarily radial, satisfying the assumption of Theorem 1.1, then there exists a sequence $t_n \rightarrow T^+(u)$, such that $u(t_n)$ admits a profile decomposition with profiles $\{\phi_j, \{x_{j,n}, \lambda_{j,n}, \xi_{j,n}, t_{j,n}\}_n\}_j$, and there is a unique compact profile, we assume it is ϕ_1 , such that*

- $\|\phi_1\|_2 \geq \|Q\|_2$,
- The associated nonlinear profile Φ_1 , is an almost periodic solution in the sense of (4.2), and it does not scatter forward nor scatter backward.

See Section 6 for a proof.

Remark 4.3. *One may compare this step to the procedure of reduction to the minimal blow up solution in the study of defocusing problem.*

Remark 4.4. *Due to the assumption (1.5), there cannot be more than one profile with mass no less than $\|Q\|_2^2$.*

4.3. Step 2: Second extraction of Profile. We need to do some further modification of profile, the following step is very standard when one wants to prove scattering type results. By arguing exactly as Section 4 of [TVZ08], we will have

Lemma 4.5. *Let Φ_1 be the nonlinear profile as in Lemma 4.2, with lifespan (T^-, T^+) . Then, according to Lemma 4.2,*

$$\Phi_1(t, x) = \frac{1}{\lambda_1^{d/2}(t)} P_t \left(\frac{x - x_1(t)}{\lambda_1(t)} \right) e^{ix\xi_1(t)}. \quad (4.3)$$

Further more, there exists $\{t_n\}_n, t_n \in (T^-, T^+)$, such that

$$P_{t_n} \xrightarrow{n \rightarrow \infty} P_0 \text{ in } L^2 \quad (4.4)$$

And the solution w to (1.1) with initial data P_0 or \bar{P}_0 satisfies

$$w(t, x) = N^{d/2}(t) L_t(N(t)(x - x(t))) e^{ix\xi(t)}. t \geq 0, N(t) \leq 1 \quad (4.5)$$

And $\{L_t\}_t$ is a precompact L^2 family.

Indeed, such a solution w , sometimes also called minimal blow up solution, already partially falls into the framework of Dodson's work [Dod12], [Dod15].

4.4. Step 3: Fast Cascade case. We exclude the so-called fast cascade, i.e. the case

$$\int_0^\infty N^3(t) < \infty. \quad (4.6)$$

In this regime, for $d = 3$, Dodson's long time Strichartz estimate, Theorem 1.24 in [Dod12] will indeed imply w is not only a L^2 solution, but an H^1 solution, see Theorem 3.13 in [Dod12], and furthermore, the energy is zero, see (3.86), (3.87) and Remark 3.14 in [Dod12]. Long time Strichartz estimate also holds for $d = 1, 2$, with extra technical difficulty, see for [Dod16], [Dod]. See Theorem 1.9 in [Dod15] for a summary.

Thus, we have

Lemma 4.6 (Dodson). *Consider w as in Lemma 4.5, assume further (4.6), then $w(0) \in H^1$ and $E(w) = 0$.*

We then have

Lemma 4.7. *Consider w as in Lemma 4.5, (4.6) cannot hold.*

Proof. The case $\|w\|_2 = \|Q\|_2$ is impossible since by Lemma 4.6, w is in H^1 and with zero energy, thus, by Lemma 2.18, $w = \frac{1}{\lambda_0^{d/2}} Q(x - x_0/\lambda_0) e^{i\gamma}$ and the solution is just a standing wave which implies $N(t) \sim 1$ and $\int_0^\infty N(t)^3 = \infty$. The case $\|Q\| < \|u\|_2 < \|Q\|_2 + \alpha$ is impossible because by Theorem 3.1, such solution must blow up in finite time. \square

4.5. Step 4: Quasisoliton case. It is in this step that we need radial assumption. Since we are considering radial solution, then w in (4.5) must also be radial, which imply that $x(t), \xi(t) \equiv 0$.

Remark 4.8. *Since all we need is $x(t), \xi(t) \equiv 0$, we may just assume u is symmetric across d linear independent planes. The observation that if u is symmetric across d linear independent planes then $x(t), \xi(t) \equiv 0$ has been pointed out by Dodson [Dod12].*

Now, we are left with the case $\int_0^\infty N^3(t) = \infty$, which is usually called Quasoliton case in the literature. We will show in this case, it must be that $\|w(0)\|_2 = \|Q\|_2$.

Lemma 4.9. *It is impossible that w is of form (4.5), $x(t), \xi(t) \equiv 0$, $\|Q\|_2 < \|w(0)\|_2 \leq \|Q\|_2 + \alpha$, and $\int_0^\infty N^3(t) = \infty$.*

And we will further show

Lemma 4.10. *Assume w is of form (4.5), $x(t), \xi(t) \equiv 0$, $\|w(0)\|_2 = \|Q\|_2$, and $\int_0^\infty N^3(t) = \infty$, then exist a sequence t_n , and parameters λ_n, γ_n such that*

$$\lim_{n \rightarrow \infty} \|\lambda_n^{d/2} w(t_n, \lambda_n x) e^{i\gamma_n} - Q\|_2 = 0. \quad (4.7)$$

Remark 4.11. *It is very natural to conjecture that the under the same assumption of Lemma 4.10, w is essentially standing wave Qe^{it} . This will be related the classification of finite time blow up solution to (1.1) with mass $\|u_0\|_2 = \|Q\|_2$. Such solutions, if one further assume the initial data is in H^1 , are completely determined by the result of Merle, [M⁺93]. At the level of L^2 , it seems to be a very hard problem.*

To understand the proof of Lemma 4.9, Lemma 4.10, one needs to understand how Dodson handles the case $\|w(0)\|_2 < \|Q\|_2$, [Dod15]. We will give a rather detailed sketch of Dodson's arguments in Section 7. Basically, one needs to use Virial identity to explore the decay of the solution and one needs to perform frequency cut-off to explore the coersiveness of energy. We will show the following:

Lemma 4.12. *Assume w is of form (4.5), $x(t), \xi(t) \equiv 0$, and $\int_0^\infty N(t)^3 = \infty$, then there exist sequences $t_n \leq T_n$, $R_n \gg \frac{1}{N(t_n)}$, $\int_0^{T_n} N^3(t) = K_n$, such that*

$$E\left(\chi\left(\frac{x}{R_n}\right) P_{\leq CK_n} w(t_n)\right) \leq \frac{1}{n} \left\| \nabla \chi\left(\frac{x}{R_n}\right) P_{\leq CK_n} w(t_n) \right\|_2^2, \quad (4.8)$$

where χ is smooth bump function localized around the origin.

We will use Lemma 4.12, Lemma 2.18 and Corollary 3.3 to deduce Lemma 4.9, Lemma 4.10.

See Section 7 for the proof of Lemma 4.12, Lemma 4.9 and Lemma 4.10.

4.6. Step 5: Approximation argument and conclusion of the proof. To conclude the proof of Theorem 1.1, we use Lemma 4.2 to reduce the dynamics of u to the unique compact profile ϕ_1 , and its associated solution Φ_1 . Then we use Lemma 4.5 to reduce the dynamic of Φ_1 to the the almost periodic solution w , and we use Lemma 4.7, Lemma 4.9 to derive that one must have

$$\|w\|_2 = \|Q\|_2, \quad \int_0^\infty N^3(t) = \infty. \quad (4.9)$$

And finally, such solution is characterized by Lemma 4.10. We show the detail in Section 8.

5. PROOF OF COROLLARY 3.3

We prove Corollary 3.3 here. Let u be as in Corollary 3.3.

First, if u is of strictly negative energy, by Lemma 2.18, we have

$$\|u\|_2 > \|Q\|_2 \quad (5.1)$$

If u is of zero energy, then (5.1) is already in the assumption of Corollary 3.3.

Thus, we can assume

$$\|u\|_2^2 \equiv \|Q + \epsilon\|_2^2 = \|Q\|_2^2 + \delta_0. \quad (5.2)$$

Note mass is a conservation law. By choosing α in Assumption 1.5, $\delta_0 \ll \int_{|x| \leq 1} |Q|^2 dx$. We will choose the $\delta(u)$ in Corollary 3.3 as $\frac{\delta_0}{2}$.

Since Q is of exponential decay, we have that, by (3.3), when t is close to $T^+(u)$ enough,

$$\langle |Q|, |\epsilon(t, x)| \rangle \ll \delta_0. \quad (5.3)$$

By (5.2), we obtain

$$\int |\epsilon|^2 \lesssim \frac{3}{4} \delta_0. \quad (5.4)$$

On the other hand, by (5.3)

$$\int_{|x| \leq 1} |Q + \epsilon|^2 \geq \delta_0/2. \quad (5.5)$$

Now fix any $A > 1$, using the trivial estimate

$$\int_{|x| \leq A} |\epsilon|^2 \lesssim e^A \int |\epsilon|^2 e^{-|x|} \quad (5.6)$$

By (3.3), and T_1 close to $T^+(u)$, we have

$$\int_{|x| \leq A} |\epsilon|^2 \ll \delta_0 \quad (5.7)$$

Thus combine (5.3), (5.7) and (5.4), we have by triangle inequality that

$$\int_{|x| \geq A} |Q + \epsilon|^2 \geq \delta_0/2. \quad (5.8)$$

Let l_0, x_0 in Lemma 3.3 be $x(T_1), \lambda(T_1)$, then the Corollary follows.

6. PROOF OF LEMMA 4.2

Lemma 4.2 should be compared with the reduction to minimal mass blow up solutions for scattering type problem. Most arguments below are standard in concentration compactness, see for example [KM06], [KM08], thus we just sketch it. We will also use a minimization procedure from [DKM15], which makes the whole proof more clear for us. We remark that we do not use the fact that u is radial here.

First, for any $\tilde{t}_n \rightarrow T^+$, up to extracting subsequence, we may assume $u(\tilde{t}_n)$ admits profile decomposition with profiles $\{\psi_j, \{\tilde{x}_{j,n}, \tilde{\xi}_{j,n}, \tilde{\lambda}_{j,n}, \tilde{t}_{j,n}\}_n\}_j$. If $\forall j$, we have $\|\psi_j\|_2 < \|Q\|_2$, then by Theorem 2.8, and the nonlinear approximation argument Proposition 2.16, one would derive that u scatters forward, which contradicts our assumption. Thus, there is at least one profile with mass no less than $\|Q\|_2^2$. On the other hand, by the asymptotically orthogonality of the mass, (2.22), and our assumption 1.5, there can only be one profile with mass no less than $\|Q\|_2$. By reordering the profile if necessary, we assume the first profile ψ_1 is the unique profile with

$$\|\psi_1\|_2 \geq \|Q\|_2. \quad (6.1)$$

Remark 6.1. *By Assumption 1.5 and the asymptotically orthogonality of mass, all other profiles has mass $\ll 1$, which implies there associated nonlinear profile is global and scattering by the small data theory, Proposition 2.5.*

Furthermore, ψ_1 must be a compact profile. Indeed, if ψ_1 is a forward scattering profile, using the nonlinear approximation Proposition 2.16, we have that u scatter forward, a contradiction. If ϕ_1 is a backward scattering profile, then we use Proposition 2.16 for the initial data $u(\tilde{t}_n)$, but run the (1.1) backwards rather than forwards, then we will get a uniform bound for $\|u\|_{L_{t,x}^{2(d+2)/d}[0,\tilde{t}_n \times \mathbb{R}^d]}$, which again implies u scatters forward, a contradiction. See [KM06] for similar arguments for energy critical wave, see also [KM08]. Since ψ_1 is a compact profile, we do not distinguish between ψ_1 and its associated nonlinear profile Ψ_1 , which is the solution to (1.1) with initial data ψ_1 .

Finally, we remark ψ_1 is not uniquely determined by the times sequence $\{\tilde{t}_n\}_n$, since one may scale or translate the profile, however, the L^2 norm of ψ_1 is uniquely determined, since L^2 is invariant under those symmetry.

To find a sequence $\{t_n\}$ such that its associated profile decomposition satisfy Lemma 4.2, we mimic the minimization procedure in Section 4 of [DKM15]. Though that paper deals with energy critical wave and energy critical Schrödinger, most arguments there are quite general and works whenever there is a satisfying profile decomposition technique.

One will need the so-called double profile decomposition at the technique level, which maybe compared to the diagonal technique which is used in the proof Arzela-Ascoli Lemma.

Lemma 6.2. *Assume $\{f_n^p\}_{n,p}$ are uniformly bounded in L^2 , assume for $\forall p$, f_n^p admits a profile decomposition with profiles $\{g_j^p\}$ such that there exists $\{\eta_j\}_j$ such that for all p ,*

$$\sum_j \eta_j < \infty, \quad \|e^{it\Delta} g_j^p\| \leq \eta_j. \quad (6.2)$$

And assume for all j , $\{e^{it\Delta} g_j^p(0)\}_p$ admits a profile decomposition with profile $h_{j,k}$, then up to extracting subsequence, there exists $n_p \rightarrow \infty$ such that $\{f_{n_p}^p\}_p$ admits a profile decomposition with profile $\{h_{j,k}\}_{j,k}$.

Remark 6.3. *According to asymptotically orthogonality of mass (2.22), we have that for all j, k , $\|h_{j,k}\|_2 \leq \|g_j\|_2$.*

Remark 6.4. *We will not need to check condition (6.2) in our work. because for all the profile decompositions involved in our work, if we reorder the profiles such that $\|g_j\| \geq \|g_{j'}\|, \forall j \geq j'$, we always have $\|g_1\|_2 \leq \sqrt{2\|Q\|_2}$, and $\|g_n\|_2 \lesssim \sqrt{\frac{1}{n-1}}$, $\forall n \geq 2$, thus (6.2) automatically holds.*

Lemma 6.2 is the natural generalization of Lemma 3.16 in [DKM15] for equation (1.1). we refer to [DKM15] for a proof. (Though the proof there is written for energy critical wave, it also works here.)

Now let us go back to the proof of Lemma 4.2.

Let \mathcal{A} be the set of the time sequence $\{\tilde{t}_n\}_n$ such that $\lim_{n \rightarrow \infty} \tilde{t}_n = T^+(u)$ and $\{u(\tilde{t}_n)\}_n$ admits a profile decomposition. Let ϕ_1 be the profile with mass no less than $\|Q\|_2^2$. Recall that ϕ_1 may not be uniquely determined by the time sequence, but $\|\phi_1\|_2$ is. We define a map for $s = \{\tilde{t}_n\}_n \in \mathcal{A}$ to \mathbb{R} as

$$\mathcal{E}(s) = \|\phi_1\|_2 = \|\Psi_1\|_2 \quad (6.3)$$

We have that (6.1) implies

$$\inf_{s \in \mathcal{A}} \mathcal{E} \geq \|Q\|_2. \quad (6.4)$$

We now claim there exists an $s_0 \in \mathcal{A}$ such that

$$\mathcal{E}(s_0) = \inf_{s \in \mathcal{A}} \mathcal{E}. \quad (6.5)$$

In fact, by Lemma 6.2, there exists $\mathcal{E}(s_p) \rightarrow \inf_{s \in \mathcal{A}} \mathcal{E}$, then apply the double profile decomposition Lemma 6.2, one will find $\{u(\tilde{t}_{n_p}^p)\}$ admits a profile decomposition and $\mathcal{E}(\{\tilde{t}_{n_p}^p\}_p) = \inf_{s \in \mathcal{A}} \mathcal{E}$.

Let us finish the proof of Lemma 4.2. Let $s_0 = \{t_n\}_n$, and $u(t_n)$ admits a profile decomposition with profiles $\{\phi_j, \{x_{j,n}, \xi_{j,n}, \lambda_{j,n}, t_{j,n}\}_n\}_j$, the associated unique nonlinear profile with mass above ground state be Φ_1 , clearly Φ_1 satisfy

$$\|Q\|_2 \leq \|\Phi_1\|_2 \leq \|Q\|_2 + \alpha. \quad (6.6)$$

We further claim Φ_1 must be an almost periodic of form (4.2). Indeed, if not, then there exists time sequence $\{a_n\}$ within the life span of Φ_1 such that $\Phi_1(a_n)$ admits a profile decomposition and any profile has mass strictly smaller than $\|\Phi_1\|_2$, then using the nonlinear approximation Proposition 2.16 and double profile decomposition Lemma 6.2, it is easy to see up to picking a subsequence $u(t_n + \lambda_1^2 a_n)$ that admits a profile decomposition and $\mathcal{E}(\{t_n + \lambda_1^2 a_n\}_n) < \mathcal{E}(s_0)$, a contradiction. This concludes the proof.

7. PROOF FOR SUBSECTION 4.5

We prove Lemma 4.12, Lemma 4.9, Lemma 4.10 here.

To prove Lemma 4.12, one needs to use the proof in [Dod15]. We do a review of the argument in [Dod15] here.

7.1. A quick review of Dodson's work [Dod15].

7.1.1. *warm up and energy tensor.* One is recommended to use the associated energy tensor to do computation. We use Einstein summation convention.

Let

$$iu_t + \Delta u = -|u|^{p-1}u. \quad (7.1)$$

(Note in our case, $p = 1 + 4/d$)

and let

$$\begin{aligned} T_{00} &= T_{00}(u) := |u|^2, \quad T_{j0} = T_{0j} = 2\Re u_j \bar{u}, \\ T_{kj} &= T_{jk} = 4\Re \bar{u}_j u_k - \delta_{kj} \Delta |u|^2 - 2\frac{p-1}{p+1} |u|^{p+1}, \end{aligned} \quad (7.2)$$

Then,

$$\partial_t T_{00} + \partial_j T_{0j} = 0, \partial_t T_{0j} + \partial_k T_{jk} = 0. \quad (7.3)$$

Let us recall Virial identity as an example. The computation here is just formal, we assume a priori all the quantities in the following computation is finite.

$$\begin{aligned} \partial_{tt} \int |x|^2 |u|^2 &= \partial_{tt} \int |x|^2 T_{00} = \partial_t \int |x|^2 \partial_t T_{00} \\ &= \partial_t \int \partial_j |x|^2 T_{j0} = \int \partial_j \partial_k |x|^2 T_{j,k} = 16E(u). \end{aligned} \quad (7.4)$$

Almost all results regarding the long time dynamic of (1.1) rely on (7.4) in some sense. Intuitively, when $E(u)$ is positive, (which is always the case when $\|u\|_2 < \|Q\|_2$), then $\int |x|^2 |u|^2$ will grow to infinity. On the other hand, since the

mass $\int |u|^2$ is conserved, this implies that mass are ejected to infinity, which should be understood as a dispersion effect.

We further summarize the last three identities in (7.4).

$$\frac{d}{dt} \int x_j T_{0j} \equiv \frac{d}{dt} \int x_j \Im u_j \bar{u} = \frac{1}{2} \int \partial_k x_j T_{jk} \equiv 4E(u). \quad (7.5)$$

7.1.2. *A sketch of Dodson's work.* Using (7.5), Dodson shows

Proposition 7.1. *it is impossible that (4.5) holds with $\int N^3(t) = \infty$ and*

$$\|w\|_2 < \|Q\|_2 - \eta, \text{ for some } \eta > 0 \quad (7.6)$$

unless $w \equiv 0$

We will use $d = 3$ here to present a sketch for the proof of Proposition 7.1. We only do the case $\xi(t), x(t) = 0$, (recall $\xi(t), x(t), N(t)$ in (4.5)).⁴

First subcase: $\xi(t) \equiv 0, x(t) \equiv 0, N(t) \sim 1$

Let T_K be the unique time such that

$$\int_0^{T_K} N^3(t) = K. \quad (7.7)$$

All the analysis below is in $[0, T_K]$ for w as in (4.5).

We first consider the subcase $\xi(t) \equiv 0, x(t) \equiv 0, N(t) \sim 1$ for w in (4.5). We remark that $x(t), \xi(t) \equiv 0$ when one only considers radial solutions. Now the solution is like a soliton, without dispersion. It is very natural to apply (7.5) to get a contradiction. However, w is not in H^1 , so one cannot directly use mass constriction (7.6) to use the coersiveness of energy and one does not have the extra integrability of xu to make sense the left side of (7.5). So, very naturally, one needs to do truncation in space and frequency.

We need a cut off version of x , for technical reason, we will need a $\psi(x)$ such that $\psi(x) = 1$ for $|x| \leq 1$, and $\psi(x) \lesssim \frac{1}{|x|}$, and $\partial_k \psi(x) x_j$ is semi positive definite⁵.

We also define the Fourier truncation, $I := I_K \equiv P_{\leq CK}$, here C is some fixed large constant. Let $F(v) := -|v|^{4/d} v$. Now note

$$(i\partial_t + \Delta)Iw = I(F(w)) = F(I(w)) + \{I(F(w) - F(Iw))\}. \quad (7.8)$$

Let $M(t)$ be the truncated version of $\int x_j \Im w_j \bar{w}$, (M denotes Morawetz action in the literature) :

$$M(t) = \int \psi\left(\frac{x}{R}\right) x_j \Im I_K w_j I_K \bar{w}. \quad (7.9)$$

Since $\int |w|^2$ is bounded, one immediately obtain that

$$|M(t)| \lesssim RK. \quad (7.10)$$

Recall (4.5), using the fact that $w = N(t)^{d/2} L_t(N(t)x)$, $N(t) \sim 1$ and $\{L_t\}$ is a precompact in L^2 , it is easy to upgrade the above to

$$|M(t)| \lesssim Ro(K). \quad (7.11)$$

⁴ Indeed, the computation is easier for $d = 1$ or $d = 2$, however, Dodson's work [Dod15] implicitly use his long time Strichartz estimate, which involves extra technical difficulty for $d = 1, 2$.

⁵ This is not hard, indeed, one first constructs some convex $f(x)$ which is like $|x|^2$ near the origin, slowly grows for $|x| \geq 2$, and take $\psi(x) x_j = \partial_j f(x)$. Note we do not need f to be uniformly strictly convex.

Now computing the derivative of $M(t)$, one has

$$\begin{aligned}
\frac{d}{dt}M(t) &= E_1 + \int \partial_k(\psi(x/R))x_j \frac{1}{2}T_{jk}(Iw) \\
&= E_1 + \int \partial_j \partial_k(\psi(x/R)x_j) \frac{1}{2} \{4\Re \bar{I}w_j Iw_k - \delta_{kj} \Delta |Iw|^2 - \frac{2 \cdot 4/d}{4/d+2} |w|^{\frac{4}{d}+2}\} \\
&= E_1 + E_2 + E_3 + 4 \int_{|x| \leq R} \left(\frac{1}{2} |\nabla u|^2 - \frac{1}{2 + \frac{4}{d}} |w|^{2+4/d} \right),
\end{aligned} \tag{7.12}$$

where E_1, E_2, E_3 are as the following.

$$E_1 = -i \int \psi\left(\frac{x}{R}\right) x (\{IF(w) - F(Iw)\} \nabla Iw + Iw \nabla \{IF(w) - F(Iw)\}) \tag{7.13}$$

$$E_2 = \int [\Delta \delta_{jk} \psi(x/R) x_j] (-\delta_{kj} |Iw|^2), \tag{7.14}$$

$$E_3 = \int_{|x| \geq R} \partial_k(\psi(x/R)x_j) \frac{1}{2} \{4\Re \bar{I}w_j Iw_k - \frac{2 \cdot 4/d}{4/d+2} |w|^{\frac{4}{d}+2}\}. \tag{7.15}$$

Here E_1 is caused by the commutator type error in (7.8).

One needs to explore the coersiveness of $\int_{|x| \leq R} \frac{1}{2} |\nabla Iw|^2 - \frac{1}{2 + \frac{4}{d}} |Iw|^6$, thus one needs to introduce an extra smooth cut-off function $\chi(x)$ which is 1 for $|x| \leq \frac{9}{10}$, and vanishes for $|x| \geq 1$. Then

$$\int_{|x| \leq R} \frac{1}{2} |\nabla Iw|^2 - \frac{1}{2 + \frac{4}{d}} |Iw|^6 \geq E(\chi(x/R)Iw) \tag{7.16}$$

We remark that strictly speaking, (7.16) is not completely right, since we neglect the error caused by the commutator $(\nabla(\chi Iw) - \chi \nabla Iw)$, since this is just a sketch, we omit this technical point, one should refer to [Dod15] for more details.

Now, using (2.33) and the important condition (7.6), one has

$$E(\|\chi(\frac{x}{R})Iw\|) \geq c_0(\eta)(\|\chi(\frac{x}{R})Iw\|^{4/d+2} + \|\nabla(\chi(x/R)Iw)\|^2). \tag{7.17}$$

Error E_2, E_3 will be estimated by

$$\begin{aligned}
|E_2| &\lesssim \frac{1}{R^2} \int_{|x| \geq R} |Iw|^2, \\
|E_3 - \int_{|x| \geq R} \partial_k(\psi(x/R)x_j) \frac{1}{2} \{4\Re \bar{I}w_j Iw_k\}| &\lesssim \int_{|x| \geq R} |Iw|^{2+4/d}
\end{aligned} \tag{7.18}$$

Note since $\partial_k(\psi x_j)$ is semipositive definite, we have

$$E_3 \geq -C_1 \int_{|x| \geq R} |Iw|^{4/d+2} \text{ for some } C_1 > 0. \tag{7.19}$$

E_1 is estimated by the commutator type estimate, see Lemma 4.7 in [Dod12].

$$\|I_K F(w) - F(I_K w)\|_{L_t^2 L_x^{2d/d+2}[0, T_K]} \lesssim o_K(1). \tag{7.20}$$

The proof of the above relies on Dodson's long time Strichartz estimate, Theorem 1.24 in [Dod12], which is indeed the key ingredient in [Dod12].

Thus,

$$\left| \int_0^{T_K} E_1 \right| \lesssim Ro_K(1) \|\nabla I_K w\|_{L^2 L_x^{2d/d-2}}. \quad (7.21)$$

And the long time Strichartz estimate will further give, see Lemma 4.5 in [Dod12],

$$\|\nabla I_K w\|_{L^2 L_x^{2d/d-2}} \lesssim K \quad (7.22)$$

Thus,

$$\int_0^{T_K} |E_1| \geq Ro_K(1)K. \quad (7.23)$$

We remark that the long time Strichartz estimate is purely analytic, does not relies on the energy structure, i.e. the difference between focusing and defocusing does not matter here.

To summarize,

$$\begin{aligned} \frac{d}{dt} M(t) &\geq 4E(\chi(x/R)Iw) + E_1 + E_2 + E_3 \\ &\geq c_0(\eta) \|\chi(x/R)Iw\|_{L^{4/d+2}} + E_1 - |E_2| - C_1 \int_{|x| \geq R} |Iw|^{4/d+2} \end{aligned} \quad (7.24)$$

In the last step, we plugged in the estimate (7.19).

Now, integrate in time on $[0, T_k]$ plug in the estimate for E_1 , (7.23), and estimate for E_2 , (7.18), we recover the estimate (3.26) [Dod15]⁶.

$$\int_0^{T_K} \frac{d}{dt} M(t) \geq \int_0^{T_K} c_0(\eta) \chi(x/R) Iw \Big\|_{L_x^{2+4/d}}^{2+4/d} - \int_0^{T_K} \sup_t \frac{1}{R^2} \|Iw(t)\|_{L_x^2}^2 - \int_0^{T_K} \int_{|x| \geq R} |Iw(t, x)|^6 - Ro_K(1)K. \quad (7.25)$$

Estimate (7.25) is enough to give a contradiction and conclude u must be zero, one is refer to [Dod15], in particular (3.26) in [Dod15] for more details. We sketch some standard arguments below for the convenience of the readers.

The left side of (7.25) is controlled by $Ro(K)$, by estimate (7.11).

On the other hand, since we assume $N(t) \sim 1$, $\int_0^{T_K} \sim \int_0^{T_K} N(t)^3 = K$, thus

$$\int_0^{T_K} \sup_t \frac{1}{R^2} \|u(t)\|_{L_x^2}^2 \lesssim \frac{1}{R^2} K \quad (7.26)$$

And, recall again u is of form (4.5), $N(t) \sim 1$, then for any time interval J of length ~ 1 , local theory of (1.1) gives $\int_J |u|_{L_x^{4/d+2}}^{4/d+2} \sim 1$. Now using the fact u is of form (7.25), i.e. all the mass of u is uniformly concentrated in physical space and frequency space, we obtain

$$\int_J |\chi(x/R)Iw|_{L_x^{4/d+2}}^{4/d+2} \sim 1, \int_J \int_{|x| \geq R} \lesssim o_R(1). \quad (7.27)$$

Thus, the right side of (7.25) is bounded below by

$$c_0(\eta)K - \tilde{C} \frac{K}{R^2} - o_R(K) - Ro(K). \quad (7.28)$$

Here \tilde{C} is some universal constant. Now one obtains that $Ro(K) \geq c_0(\eta)K - \tilde{C} \frac{K}{R^2} - o_R(K) - Ro(K)$, a contradiction.

⁶the numerics here are slightly different, because in [Dod15], that part is done for $d = 1$, and the error caused by E_1 is neglected in this step but treated later.

The key point to conclude a contradiction by using $Ro(K) \geq c_0(\eta)K - \tilde{C}\frac{K}{R^2} - o_R(K) - Ro(K)$ is that here $c_0(\eta)$ in (7.17) does not depend on R or K , so one can first choose R large enough, then one further choose K large enough to get a contradiction.

General Case: $\xi(t), x(t) \equiv 0$

Now, for general case with $\xi(t), x(t) = 0$, (which covers all general radial solution), it is very natural to define the Morawetz action as

$$M(t) := \int \psi \left(\frac{xN(t)}{R} \right) xN(t) \Im \nabla I w I \bar{w}. \quad (7.29)$$

However, if one directly relies on the above Morawetz action to argue as previously, one will face the problem that one does not have good control about $N'(t)$. This is handled by Dodson using so called "upcoming algorithm", basically he constructs some slowly oscillating $\tilde{N}(t) \lesssim N(t)$ according to the behavior of $N(t)$, and constructs Morawetz action as

$$M(t) = \int \psi \left(\frac{x\tilde{N}(t)}{R} \right) x\tilde{N}(t) \Im \nabla I w I \bar{w} \quad (7.30)$$

The proof left follows is in principle as the previous subcase where $N(t) \sim 1$, see section 4 in [Dod15] for more details.

As emphasized in the end of the previous case, the key part and the only part Dodson's proof using the fact $\|u\|_2 < \|Q\|_2 - \eta$ is that this will give a universal constant $c_0(\eta)$ such that

$$E(\chi(\tilde{N}(t)x/R)Iw) \geq c_0(\eta)[\|\nabla \chi(\tilde{N}(t)x/R)Iw\|_2^2 + \|\chi(\tilde{N}(t)x/R)Iw\|_{L_x^{4/(d+2)}}^{4/(d+2)}]. \quad (7.31)$$

7.2. Proof of Lemma 4.12. The proof of Lemma 4.12 is by contradiction. Indeed, if Lemma 4.12 does not hold, then one recovers (7.31) even the mass of w is not under the ground state. Then, one argue as the proof of Proposition 7.1, which we just reviewed in the previous subsection, to conclude $w = 0$, which is clearly a contradiction since $\|w\|_2 \geq \|Q\|_2$.

Remark 7.2. Proposition 7.1 holds for general non radial solution by using a version of interaction Morawetz estimate. Ever since [CKS⁺08], there are a lot of works using interaction version of certain estimates to show results for general solutions rather than radial solution, such as [Dod12], [Dod15] and many others. However, we cannot have a natural useful generalization of Lemma 4.12 here. It seems to us if one directly follows the arguments in [Dod15], where a nonradial version Proposition 7.1 is proved, one can only conclude that there exists a sequence $t_n \leq T_n$, $R_n \gg \frac{1}{N(t_n)}$, ξ_n, x_n , $\int_0^{T_n} N^3(t) = K_n$, such that

$$E(\chi(\frac{x-x_n}{R_n})P_{\leq CK_n}e^{-i\xi_n x}w(t_n)) \leq \frac{1}{n}\|\nabla \psi(\frac{x-x_n}{R_n})P_{\leq CK_n}e^{-i\xi_n x}w(t_n)\|_2^2 \quad (7.32)$$

Formula (7.32) is of no use to us, because we do not have control of x_n here, thus we are not ensured $\chi(\frac{x-x_n}{R_n})P_{\leq CK_n}e^{-i\xi_n x}w(t_n)$ contains almost all the mass of w as $n \rightarrow \infty$, which will be critical later.

7.3. Proof of Lemma 4.10. Lemma 4.10 is implied by Lemma 4.12, with the help of Lemma 2.19. We present a short proof here. Recall w is of form (4.5), and since we are consider radial solution, we have $\xi(t) = 0, x(t) = 0$. Since $R_n \gg \frac{1}{N(t_n)}$, $K_n \gg 1$ and $\{L_t\}_t$ is a precompact L^2 family, we easily have

$$\|w(t_n) - \chi(x/R_n)P_{<CK_n}w(t_n)\|_{L^2} \xrightarrow{n \rightarrow \infty} 0. \quad (7.33)$$

Now let $\tilde{w}_n := \chi(x/R_n)P_{<CK_n}w(t_n)$. Clearly, to prove Lemma 4.10, we only need to show there exists λ_n, γ_n such that

$$\lambda_n^{d/2}w_n(\lambda_n x)e^{i\gamma_n} \rightarrow Q \text{ in } L^2. \quad (7.34)$$

But (7.34) follows from Lemma 2.19 because

$$\begin{aligned} \|w_n\| &\leq \|w\|_2 \leq \|Q\|_2 + \frac{1}{n}, \\ E(w_n) &\leq \frac{1}{n}\|\nabla w_n\|_2^2 \end{aligned} \quad (7.35)$$

7.4. Proof of Lemma 4.9. Now we turn to the proof of Lemma 4.9. Note we restrict ourselves to radial solutions. Let t_n, K_n, R_n be as in Lemma 4.12, let $v_n = \chi(x/R_n)P_{\leq CK_n}u(t_n)$, $\lambda_n = \frac{\|\nabla Q\|_2}{\|\nabla v_n\|_2}$. We apriori have

$$\|v_n\|_2 \leq \|Q\|_2 + \alpha, \quad \alpha \text{ small enough}, \quad (7.36)$$

Thus, when n is large enough, $1/n \leq \alpha$, thus by Lemma 2.19, we can find a sequence of γ_n , such that

$$\left\| \frac{1}{\lambda_n^{d/2}}v_n\left(\frac{x}{\lambda_n}\right)e^{i\gamma_n} - Q \right\|_{H^1} \leq \delta(\alpha) \leq 1. \quad (7.37)$$

Note the space translation parameter in Lemma 2.19 will not appear because our functions are all radial.

Also recall $w(t) = N(t)^{d/2}L_t(N(t)x)$, L_t is a precompact L^2 family, $N(t) \leq 1$, thus the condition $K_n \gg 1, R_n \gg \frac{1}{N(t_n)}$ implies

$$\|w(t_n) - v_n\|_2 = o_n(1) \quad (7.38)$$

and further it implies that $\left\{ \frac{1}{N^{d/2}(t_n)}v_n\left(\frac{x}{N(t_n)}\right) \right\}_n$ is a precompact L^2 family.

Thus, by (7.37) we have $N(t_n) \sim \lambda_n$. Now let $f_n = \left\{ \frac{1}{N^{d/2}(t_n)}v_n\left(\frac{x}{N(t_n)}\right) \right\}_n$, we have that

- (1) f_n is a precompact L^2 sequence.
- (2) f_n is uniformly bounded.
- (3) $E(f_n) \lesssim \frac{1}{n}$. (This contains the possibility that $E(f_n)$ is negative).

Up to extracting a subsequence, we may assume f_n strongly converges to f_0 in L^2 , in particular, with (7.38), we obtain

$$\|f_0\| = \|w\|_2, \|Q\| < \|f\|_2 < \|Q\|_2 + \alpha \quad (7.39)$$

Now, by Fatou's Lemma, $\|f_0\|_{\dot{H}^1} \leq \liminf_n \|f_n\|_{\dot{H}^1}$.

By interpolation⁷ between L^2 and H^1 , $\|f_n\|_{L^{4/d+2}} \rightarrow \|f_0\|_{L^{4/d+2}}$.

Thus, we have the condition

$$E(f_0) \leq \liminf E(f_n) \leq 0 \quad (7.40)$$

Now, to summarize, with (7.38), strong convergence of f_n to f_0 in L^2 , we have

⁷Note we do not need the radial Sobolev embedding here

- w is an almost periodic solution,

$$w(t) = N^{d/2}(t)L_t(N(t), x), t \geq 0, N(t) \leq 1.$$

- L_{t_n} converges strongly in L^2 to f .
- f is in H^1 , and if of nonpositive energy.

These three property is enough to derive a contradiction, we prove something slightly more general for the convenience of future work.

The proof we are left with will not depend on that the radial property, we only need that w is almost periodic in the sense that

$$w(t) = \frac{1}{\lambda^d(t)} L_t\left(\frac{x - x(t)}{\lambda(t)}\right) e^{ix\xi(t)}, t \geq 0. \quad (7.41)$$

Let F be the solution to (1.1) with initial data f , then by Theorem 3.1, F will blow up in finite time $T^+ > 0$, and according to Corollary 3.3, there exists $\delta_0 = \delta_0(F) > 0$, such that for $\forall A > 0$, there exists $T_A < T^+$, $x_0 \in \mathbb{R}^d$

$$\int_{|x-x_0| \leq l_0} |F(T_A, x)|^2 \geq \delta_0, \int_{|x-x_0| \geq Al_0} |F(T_A, x)|^2 \geq \delta_0. \quad (7.42)$$

And note by standard local theory of (1.1),

$$\|F(t, x)\|_{L_{t,x}^{2(d+2)/d}([0, T_A] \times \mathbb{R}^d)} \leq C_A < \infty. \quad (7.43)$$

Remark 7.3. Note here δ_0 is fixed once w is fixed, and A can be chosen arbitrary large (by choosing $T = T_A$ close to T^+ enough.)

On the other hand, since w is an almost periodic solution of form (7.41), we claim

Lemma 7.4. Assume w is of form (7.41), then, for any $\delta > 0$, there exists $A = A_\delta$ such that if for some $z_0 \in \mathbb{R}^d$, $t_0 \in \infty$, $h_0 > 0$ such that

$$\int_{|x-z_0| \leq h_0} |w(t_0)|^2 \geq \frac{\delta}{2}, \quad (7.44)$$

then we must have

$$\int_{|x-z_0| \geq Ah_0} |w(t_0, x)|^2 dx \leq \frac{\delta}{2}. \quad (7.45)$$

Lemma (7.4) will be proven in Subsection 7.5, let us assume it at the moment and finish the proof of Lemma 4.9. Let us fix $\delta_0 = \delta_0(w)$, and picking $\delta_1 = \delta_0/2$, and let $A = A_{\delta_1}$ as in Lemma 7.4 such that (7.44) holds, and as mentioned before, we can use Corollary 3.3 to find $T = T_A$ such that (7.42) holds and we emphasize again (7.43) holds.

Now since $L_{t_n} \rightarrow F(0)$ in L^2 , then for $\forall \epsilon > 0$, there exists $n_0 = n_0(\epsilon)$ such that

$$\|L_{t_{n_0}} - F(0)\|_2 < \epsilon \quad (7.46)$$

Using the stability argument, Proposition 2.7, by choosing ϵ small enough, (according to C_A), then (1.1) with initial data $L_{t_{n_0}}$ has a solution, we call it v , which exists in $[0, T_A]$, such that

$$\|v(T_A) - w(T_A)\|_2 \leq \delta_0/10. \quad (7.47)$$

By (7.42) and triangle inequality, we have

$$\int_{|x-x_0| \leq l_0} |v(T_A, x)|^2 \geq \delta_0/2, \quad \int_{|x-x_0| \geq Al_0} |w(T_A, x)|^2 \geq \delta_0/2. \quad (7.48)$$

On the other hand, since $v(t)$ solves (1.1) in $[0, T_A]$ with initial data $L_{t_{n_0}}$, and $w(t)$ solves (1.1) with $w(t_{n_0}) = \frac{1}{\lambda^{d/2}(t_{n_0})} L_{t_{n_0}} \left(\frac{x - x(t_{n_0})}{\lambda(t_{n_0})} \right) e^{ix\xi(t_{n_0})}$, by the local well posedness theory (uniqueness of the solution), we have w is defined in $[t_{n_0}, t_{n_0} + \lambda(t_{n_0})^2 T_A]$, and

$$\begin{aligned} & w(t_{n_0} + \lambda(t_{n_0})^2 T_A, x) \\ &= \frac{1}{\lambda(t_{n_0})^{d/2}} v \left(T_A, \frac{x - x(t_{n_0}) - 2\xi(t_{n_0})\lambda(t_{n_0})^2 T_A}{\lambda(t_{n_0})} \right) e^{-i\lambda(t_{n_0})^2 |\xi(t_{n_0})|^2 T_A} e^{ix\xi(t_{n_0})} \\ &= \frac{1}{\lambda(t_{n_0})^2} v \left(T_A, \frac{x - \tilde{x}_0}{\lambda(t_{n_0})} \right) e^{i\gamma_0} e^{i\xi(t_{n_0})x}. \end{aligned} \quad (7.49)$$

where $\tilde{x}_0 = x(t_{n_0}) + 2\xi(t_{n_0})\lambda(t_{n_0})^2 T_A \lambda(t_{n_0})$.

Let $\tilde{t}_{n_0} := t_{n_0} + \lambda(t_{n_0})^2 T_A$, $z_0 = \tilde{x}_0 + x_0$, and plug in (7.48), we obtain

$$\int_{|x - z_0| \leq \lambda(t_{n_0})l_0} |w(\tilde{t}_{n_0}, x)|^2 \geq \delta_0/2, \quad \int_{|x - z_0| \geq A\lambda(t_{n_0})l_0} |w(\tilde{t}_{n_0}, x)|^2 \geq \delta_0/2 \quad (7.50)$$

This contradicts Lemma 7.4. To finish the proof of Lemma 4.7, we are left with the proof of Lemma 7.4, which will be done in the following subsection.

7.5. Proof of Lemma 7.4. Indeed, we only need to prove Lemma 7.4 for $w(t) = L_t$, $t \geq 0$. Since the statement of Lemma already takes space translation into account, and the l_0 takes care of the scaling, and the phase $e^{ix\xi(t)}$ plays no role in this argument.

Thus, we reduce the proof to following Lemma

Lemma 7.5. *For a precompact L^2 family $\{L_t\}_{t \geq 0}$, (not necessarily radial), $\forall \delta > 0$ there exists $A > 0$ such that if for some $l_0 > 0$,*

$$\int_{|x| \leq l_0} |L_{t_0}|^2 \geq \delta, \quad (7.51)$$

then

$$\int_{|x| \geq Al_0} |L_{t_0}|^2 \leq \delta. \quad (7.52)$$

Proof. Since $\{L_t\}$ is precompact in L^2 , by standard approximation argument, one may without loss of generality may assume L^t is uniformly bounded and their support are uniformly compact. Thus (7.51) implies $1 \lesssim l_{t_0}$, since L_t is uniformly bounded. Thus, when A is large enough, clearly (7.52) holds. \square

8. PROOF OF THEOREM 1.1

Let Φ_1 , w be as in Lemma 4.5. Recall we have that $\{u(t_n)\}$ admits profile decomposition with profiles $\{\phi_j, \{x_{j,n}, \lambda_{j,n}, \xi_{j,n}, t_{j,n}\}_n\}_j$. Since now we only consider radial solution, we indeed have $x_{j,n} = \xi_{j,n} = 0$.

And as explained in Subsection 4.6 we have

$$\|w\|_2 = \|Q\|_2, \quad \int N(t)^3 = \infty \quad (8.1)$$

Thus, by Lemma 4.10, there exists a time sequence s_n and parameter $\{\lambda_n, \gamma_n\}_n$ such that

$$\lim_{n \rightarrow \infty} \|\lambda_n^{d/2} w(s_n, \lambda_n x) - Q e^{-i\gamma_n}\|_2 = 0. \quad (8.2)$$

We point out there is a slight abuse of notation in Lemma 4.2, Lemma 4.5, and Lemma 4.10. The time sequences $\{t_n\}_n$ in these lemmas are not necessarily same. From now on, we change the time sequence in Lemma 4.10 to $\{s_n\}_n$, and change the time sequence in Lemma 4.5 to $\{l_n\}$.

Note by extracting subsequence, we may without loss of generality assume the γ_n in (8.2) satisfy $\gamma_n \equiv \gamma_0$. Note phase symmetry is a compact symmetry in L^2 .

We claim (8.2) implies

Lemma 8.1. *There exists a time sequence $\{h_n\}$ such that $\Phi_1(h_n)$ admits a profile decomposition with profiles $\{v_j\}_j$. (We will not track the associated parameters here.) Moreover, there is a profile, we call it v_1 such that $v_1 = Qe^{-i\gamma_0}$*

Remark 8.2. *Since $\|\Phi_1\|_2 = \|w\|_2 = \|Q\|_2$, this indeed implies v_1 is the only profile and one can conclude similar strong convergence results as in (8.2).*

Proof of Lemma 8.1. Recall P_{l_n}, P_0, w, Φ in Lemma 4.5. Recall we have already changed the notation t_n to l_n in Lemma 4.5. Without loss of generality, we assume in Lemma 4.5, the initial data of w is P_0 . Indeed, since $P_{l_n} \rightarrow P_0$, this implies $\Phi(l_n)$ admits a profile decomposition with only one profile P_0 , and with stability argument, Proposition 2.7 or Proposition 2.16, it further implies $\{\Phi(l_n + \lambda_1(l_n^2)s_p)\}$ admits profile decomposition $w(s_p)$ for each p , also note $\{w(s_p)\}$ admits a profile decomposition with only one profile $Qe^{i\gamma_0}$ by (8.2), thus by Lemma 6.2, there exists a sequence $\{n_p\}_p$, such that $\Phi_1(l_{n_p} + \lambda(l_{n_p})^2s_p)$ admits a profile decomposition, one of the profile is $Qe^{i\gamma}$. \square

To finish the proof of Theorem 1.1, let h_p be as in Lemma 8.1, one simply argues as in the proof of Lemma 8.1 we just did, and use Lemma 6.2 to do double profile decomposition for $\{u(t_n + \lambda_{1,n}^2h_p)\}_{n,p}$.

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APPENDIX A. PROOF OF THEOREM 1.4

As shown in the proof of Lemma 4.2, such solution u is of form

$$u(t, x) = \frac{1}{\lambda^{d/2}(t)} V_t\left(\frac{x - x(t)}{\lambda(t)}\right) e^{ix\xi(t)}. \quad (\text{A.1})$$

$\{V_t\}$ is a compact L^2 family and $\|V_t\|_2 = \|Q\|_2$. We assume u blows up in finite time T . To prove Theorem 1.4, we need only exclude the possibility

$$\lambda(t) \gtrsim (T - t)^{2/3-}. \quad (\text{A.2})$$

Assume (A.2) holds, do a (inverse) pseudo conformal transformation of u , we will get a solution v such that

$$v(\tau, y) = \frac{1}{\tau^{d/2}} u\left(T - \frac{1}{\tau}, \frac{y}{\tau}\right) e^{i\frac{1}{4}\frac{|y|^2}{\tau}} = N^{d/2}(\tau) W_\tau(N(\tau)y + y(\tau)) e^{i\tilde{\xi}(\tau)y} \quad (\text{A.3})$$

Where

$$\begin{aligned} N(\tau) &= \frac{1}{\tau} \frac{1}{\lambda(T - \frac{1}{\tau})}, \\ W_\tau(z) &= V_{T - \frac{1}{\tau}}(z) e^{i\frac{1}{4}\frac{1}{N^2(\tau)\tau}|z|^2} e^{-i(x(T - \frac{1}{\tau}))^2\tau}, \\ y(\tau) &= \tau x(T - \frac{1}{\tau}), \tilde{\xi}(\tau) = \frac{\xi(T - \frac{1}{\tau})}{\tau} + 2x(T - \frac{1}{\tau}) \end{aligned} \quad (\text{A.4})$$

Note the exact value of $y(\tau), \tilde{\xi}(\tau)$ actually do not matter.

We obtain by (A.2)

$$N(\tau) \leq \frac{1}{\tau} \lesssim \frac{1}{\tau} \tau^{2/3-} \lesssim \left(\frac{1}{\tau}\right)^{1/3+} \quad (\text{A.5})$$

On the other hand, one has the scaling lower bound

$$\lambda(t) \lesssim (T - t)^{1/2} \quad (\text{A.6})$$

Thus, $N(\tau) \geq \tau^{-1/2}$,

This further implies W_τ is also precompact in L^2 , since multiplying $e^{i\frac{1}{4}\frac{1}{N^2(\tau)\tau}|z|^2}$ is a compact perturbation. This contradicts Lemma 4.7, since now we have $\int_0^\infty N^3(\tau) < \infty$.

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